Representations of the deformed U(su(2)) and U(osp(1,2)) algebras [1]

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Abstract

The polynomial deformations of the Witten extensions of the U(su(2)) and U(osp(1,2)) algebras are three generator algebras with normal ordering, admitting a two generator subalgebra. The modules and the representations of these algebras are based on the construction of Verma modules, which are quotient modules, generated by ideals of the original algebra. This construction unifies a large number of the known algebras under the same scheme. The finite dimensional representations show new features such as the multiplicity of representations of the same dimensionality, or the existence of finite dimensional representations only for some dimensions.

[1]Talk given by C. Daskaloyannis, at the Barut memorial conference, Edirne - Turkey, Dec. 1995

1 Introduction

Quantum algebras [1] are recently attracting much attention, both because of their rich mathematical structure and their potential applications in physics. The introduction of the q-deformed harmonic oscillator [2, 3, 4] has been soon thereafter followed by the introduction of the generalized deformed oscillator [5, 6, 7]. The idea of the generalized oscillator was also introduced in mathematics twenty years ago, under a totally different context and using a different language by Joseph[8] one year after the introduction of the Q- oscillator algebra by Arik and Coon[9]

In a similar way the study of the quantum algebras su$_q$(2) and su$_q$(1,1) [2, 3, 10] has been followed by the introduction of generalized deformed versions of su(2) and su(1,1). The first generalized versions of su(2) in physics were introduced by Roček [11] and Polychronakos [12]. It should be noticed that the same problem has been independently studied (using a slightly different language) in a mathematical framework by Smith [13]. The "generalized versions of su(2)" of physics publications [11, 12] are termed in mathematical publications [13] as "algebras similar to the enveloping algebra of sl(2)". The motivation of the physics investigations was initially the need to find new models for the description of the symmetries of various physical systems, while in mathematics the goal was mainly the construction of new examples of noncommutative noetherian rings (see also related references in Lenczewski[14]).

A more general framework for nonlinear deformations of U(su(2)) and U(su(1,1)) has been introduced by Delbecq and Quesne [15, 16, 17] as a generalization of the Witten[18] algebras.

The study of the representation theory of these nonlinear algebras is in general an open problem, presenting features which are not present in the case of the corresponding Lie algebras or superalgebras. The main new features are the possibility for the existence of multiple modules of the same dimensionality, as well as the selective existence of modules for some dimensionalities. These variations reveal additional difficulties on the realization of the Hopf algebra structure at these algebras.

Sun and Li[19] studied the representation theory of the deformed U(su(2)) algebra of Roček[11] and Polychronakos[12], while from the rich variety of deformations introduced by Delbecq and Quesne[15] only the special cases of the algebras $\mathcal{A}(2,1)$ [16] and $\mathcal{A}(3,1)$ [17] have been considered in some detail. In parallel, special deformations of U(su(2)) and U(su(1,1)) possessing representation theories as close as possible to the usual U(su(2)) and U(su(1,1)) algebras have been introduced and their possible applications in physics considered [20, 21].

The study of the representations of the polynomial deformations of the U(sl(2))

(or U(su(2)) or U(su(1,1))), given by Smith[13], is based on the construction of Verma modules which are quotient modules, generated by simple ideals of the original algebra. This construction was applied independently by Sun and Li[19] at the same algebras. This method was applied to the polynomial deformations of U(osp(1,2)) by Van der Jeugt and Jagannathan[22]. The same method (but using different terminology) was previously applied by Gruber *et al.*[23] to the U(su(2)) algebra and by Doebner *et al.*[24] to the U(su(1,1)) algebra, and by Gruber and Smirnov[25] to the deformed U_q(su(2)) and U_q(su(1,1)). The scope of the present presentation is to give a general method for the study of the representation theory of all nonlinear deformed U(su(2)) and U(su(1,1)) algebras introduced by Delbecq and Quesne[15, 16, 17], and to their U(osp(1,2)) counterparts by appropriately generalizing the method used by Smith[13] and Van der Jeugt and Jagannathan[22].

2 Verma modules of the deformed su(2) algebras

Let us consider the enveloping algebra $\mathcal{A}$

$$\mathcal{A} = \mathbf{C}[J_+, J_-, J_0], \tag{1}$$

generated by all the polynomial combinations of the generators $J_\pm, J_0, 1$, which satisfy the relations:

$$J_0 J_+ = J_+ G(J_0), \tag{2}$$
$$J_- J_0 = G(J_0) J_-, \tag{3}$$
$$J_- J_+ = s J_+ J_- + f(J_0), \tag{4}$$

where s is a complex number and $G(z)$, $f(z)$ are polynomials of order λ and μ respectively:

$$G(z) = \sum_{n=0}^{\lambda} g_n z^n, \quad f(z) = \sum_{n=0}^{\mu} f_n z^n. \tag{5}$$

In the special case of $s = 1$ the algebras are deformed generalizations of the U(sl(2)) [13] (or U(su(2)) or U(su(1,1)) [11, 12]) algebras. In the case of $s = -1$ the above defined algebras are generalizations of the U(osp(1,2)) algebra [22].

The existence of the rules (2), (3) and (4) imply that the algebra A accepts a normal ordering $J_+ \prec J_0 \prec J_-$. (In physics the notion of the normal ordering corresponds to the notion of the lexicographic order used in the mathematical publications.) This means that:

Proposition 1 *The algebra $\mathcal{A}$ is a vector space with a basis of the monomials:*

$$(J_+)^m\,(J_0)^n\,(J_-)^p\,, \qquad m,n,p \in \mathbf{N}, \tag{6}$$

or

$$\mathcal{A} = \mathrm{Span}\left\{(J_+)^m\,(J_0)^n\,(J_-)^p \mid m,\, n,\, p \in \mathbf{N}\right\},$$

where $\mathbf{N}$ is the set of all natural numbers plus the zero.

The rule (2) means that the algebra:

$$\mathcal{B}_+ = \mathbf{C}\,[J_+,\; J_0]$$

is a subalgebra of the original algebra $\mathcal{A}$ with normal ordering, i.e.

$$\mathcal{B}_+ = \mathrm{Span}\left\{(J_+)^m\,(J_0)^n \mid m,\, n \in \mathbf{N}\right\}.$$

Let us define the left principal [26][ch. I, §8.7] ideal

$$\mathcal{W}_+ = \mathcal{A}J_- = (J_-)\,, \tag{7}$$

and the quotient space

$$\mathcal{T}_+ = \mathcal{A}/\mathcal{W}_+. \tag{8}$$

The regular projection π_+

$$\mathcal{A} \xrightarrow{\pi_+} \mathcal{T}_+$$

is defined by:

$$\pi_+(A) = A \bmod \mathcal{W}_+, \qquad \forall A \in \mathcal{A}.$$

The base of the vector space $\mathcal{T}_+$ is given by:

$$F(m,n) = \pi_+\left((J_+)^m\,(J_0)^n\right) \stackrel{\mathrm{def}}{=} (J_+)^m\,(J_0)^n \mod \mathcal{W}_+. \tag{9}$$

There is a bijection from the vector space $\mathcal{T}_+$ onto the subalgebra $\mathcal{B}_+ \subset \mathcal{A}$, defined as follows:

$$\mathcal{T}_+ \;\stackrel{i}{\longleftrightarrow}\; \mathcal{B}_+ = \mathbf{C}[J_+, J_0], \tag{10}$$

and

$$\mathcal{T}_+ \ni \pi_+(J_+^m J_0^n) \stackrel{i}{\longleftrightarrow} J_+^m J_0^n \in \mathcal{B}_+.$$

Let $A \in \mathcal{A}$, $b = \pi_+(B)$, and μ_+ be the mapping:

$$\mathcal{A} \ni A \xrightarrow{\mu_+} \mu_+(A) \in \mathrm{End}(\mathcal{T}_+),$$

where $\mathrm{End}(\mathcal{T}_+)$ is the set of the linear transformations defined on $\mathcal{T}_+$. This mapping is given by:

$$\mu_+(A)b = \pi_+(A\pi_+^{-1}(b)) = \pi_+(AB).$$

We can verify indeed that the above mapping is a left module, which is called quotient module [26][ch. II, §1.3, example(6)] of the algebra $\mathcal{A}$. This fact can be verified by calculating the action of the generators on the base (9). This procedure, however, leads to complicated equations not necessary for our purposes.

The next step is to consider the left principal ideal of the algebra $\mathcal{B}_+$

$$\mathcal{W}_+(\eta) = \mathcal{B}_+(J_0 - \eta), \tag{11}$$

and the quotient space

$$\mathcal{T}_+(\eta) = \mathcal{B}_+/\mathcal{W}_+(\eta). \tag{12}$$

In a way similar to the previous one we define the regular projection π_η

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi_+} & \mathcal{T}_+ \\ {\scriptstyle\pi}\big\downarrow & & \big\updownarrow{\scriptstyle i} \\ \mathcal{T}_+(\eta) & \xleftarrow{\pi_\eta} & \mathcal{B}_+ \end{array}$$

Let us define:

$$\pi = \pi_\eta \circ i \circ \pi_+.$$

The base of the vector space $\mathcal{T}_+(\eta)$ is given by:

$$F(\eta, m) = \pi\left((J_+)^m\right) \stackrel{\text{def}}{=} (J_+)^m \mod \mathcal{W}_+(\eta). \tag{13}$$

The vector space $\mathcal{T}_+(\eta)$, which is a quotient module of the algebra $\mathcal{B}_+$, defines a left module of the original enveloping algebra $\mathcal{A}$. Let $A \in \mathcal{A}$, $b = \pi(B)$, and μ be the mapping:

$$\mathcal{A} \ni A \xrightarrow{\mu} \mu(A) \in \mathrm{End}(\mathcal{T}_+(\eta)),$$

which is given by:

$$\mu(A)b = \pi(A\pi^{-1}(b)) = \pi(AB).$$

The above mapping defines a left module of the algebra $\mathcal{A}$. We must point out that the vector space $\mathcal{T}_+(\eta)$ is isomorphic to the subalgebra $\mathcal{C} \subset \mathcal{B}_+ \subset \mathcal{A}$ defined by the base J_+^m:

$$\mathcal{T}_+(\eta) \xleftrightarrow{j} \mathcal{C} = \mathbf{C}\left[J_+\right].$$

After calculating the action of the generators on the base (13) we can prove the following proposition:

Proposition 2 *The enveloping algebra A, the generators of which satisfy the following relations:*

$$
\begin{aligned}
J_0 J_+ &= J_+ G(J_0), \\
J_- J_0 &= G(J_0) J_-, \\
J_- J_+ &= s J_+ J_- + f(J_0),
\end{aligned}
$$

has a left module $\mathcal{T}_+(\eta)$ such that:

$$
\begin{aligned}
\mu(J_+) F(\eta, m) &= F(\eta, m+1), \\
\mu(J_0) F(\eta, m) &= G^{[m]}(\eta) F(\eta, m), \\
\mu(J_-) F(\eta, m) &= \Phi(\eta, m) F(\eta, m-1),
\end{aligned}
\tag{14}
$$

where:

$$
G^{[m]} = \underbrace{G \circ G \circ G \circ \ldots \circ G}_{m(times)}, \quad G^{[0]}(z) = \mathrm{Id}(z) = z, \tag{15}
$$

and

$$
\Phi(\eta, 0) = 0, \quad \Phi(\eta, m) = \sum_{k=1}^{m} s^{k-1} f\left(G^{[m-k]}(\eta)\right). \tag{16}
$$

This representation corresponds to a "minimum weight" module, because:

$$
\mu(J_-) F(\eta, 0) = 0. \tag{17}
$$

Using the same methods as in ref. [15] the "Casimir" operator can be defined:

$$
C = J_+ J_- + \rho(J_0) \tag{18}
$$

where the function $\rho(z)$ satisfies the "consistency" equation:

$$
s\rho(z) - \rho(G(z)) = f(z). \tag{19}
$$

The "Casimir" operator satisfies the following equations:

$$
[C, J_0] = 0, \qquad [J_-, C]_s = 0, \qquad [C, J_+]_s = 0,
$$

where

$$
[A, B]_q = AB - qBA.
$$

We can show the following relation:

$$\mu(C)F(\eta, m) = s^m \rho(\eta) F(\eta, m). \tag{20}$$

This is the representation in which both J_0 and the "Casimir" operator C are diagonal.

In the special case of $s = 1$ the algebras are deformed generalizations of the U(sl(2)) (or U(su(2)) or U(su(1,1))) algebra, the operator C being indeed the Casimir operator of the algebra. In the case of $s = -1$ the above defined algebras are generalizations of the U(osp(1,2)) algebra, the operator C satisfying the relations:

$$[C, J_0] = 0, \quad \{C, J_-\} = 0, \quad \{C, J_+\} = 0.$$

One can then define the operator

$$C_2 = C^2 = \left(J_+ J_- + \rho(J_o)\right)^2, \tag{21}$$

which is the Casimir operator of the algebra:

$$[C_2, J_0] = 0, \quad [J_-, C_2] = 0, \quad [C_2, J_+] = 0.$$

Proposition 3 *In the general case $s = \mathrm{e}^{2i\pi/k}$, $k \in \mathbf{N}$, we can find a Casimir operator $\mathcal{D}$, which is a function of the operator C:*

$$\mathcal{D} = C^k \tag{22}$$

The eigenvalues of the above Casimir are calculated using eqs (20) and (22)

$$\mu\left(\mathcal{D}\right) F\left(\eta, m\right) = \rho^k(\eta) F\left(\eta, m\right).$$

Using the properties of the dual algebra $\mathcal{A}^*$ we can prove the following proposition:

Proposition 4 *Let η_N be a root of the function $\Phi(\eta, N)$ for some natural number N such that:*

$$\Phi(\eta_N, N) \;=\; 0, \tag{23}$$

and

$$\Phi(\eta_N, m) \;\neq\; 0, \quad \textit{for} \quad m = 1, 2, \ldots, N-1$$

then the $\mathcal{T}_+(\eta_N)$ is a N-dimensional left $\mathcal{A}$-module and its dual $\mathcal{T}^*(\eta_N)$ is a right $\mathcal{A}$-module. In this case the $\mu(J_0)$ defined by equation (14) is a $N \times N$ matrix satisfying the Cayley-Hamilton equation:

$$\mu\left((J_0 - \eta) \cdot (J_0 - G(\eta_N)) \cdot \left(J_0 - G^{[2]}(\eta_N)\right) \cdots \left(J_0 - G^{[N-1]}(\eta_N)\right)\right) \;=\; 0 \tag{24}$$

and

$$\mu\left(J_\pm^N\right) = 0 \tag{25}$$

Each algebra is then characterized by the functions $G(z)$, $f(z)$ and the constant s. Starting from these elements we can construct the representation of the algebra and we can calculate the functions $\Phi(\eta, m)$ and $\rho(\eta)$ and we find a normalized basis:

$$\phi(\eta, m) = \frac{1}{\sqrt{[\eta, m]!}} F(\eta, m), \tag{26}$$

where:

$$[\eta, m] = |\Phi(\eta, m)| = |\, [[\eta, m]] \,|$$
$$[\eta, 0]! = 1, \quad [\eta, m]! = [\eta, m]\, [\eta, m-1]!.$$

The "normalized" basis satisfies the following relations:

$$\begin{aligned}
&\mu(J_+)\phi(\eta, m) = \sqrt{[\eta, m+1]}\phi(\eta, m+1), \\
&\mu(J_0)\phi(\eta, m) = G^{[m]}(\eta)\phi(\eta, m), \quad \mu(C)\phi(\eta, m) = s^m \rho(\eta)\phi(\eta, m), \\
&\mu(J_-)\phi(\eta, m) = \mathrm{sign}\,(\Phi(\eta, m))\,\sqrt{[\eta, m]}\phi(\eta, m-1) = \\
&= \frac{|\, [[\eta, m]] \,|}{\sqrt{[\eta, m]}}\phi(\eta, m-1),
\end{aligned} \tag{27}$$

where $\mathrm{sign}\,(x)$ is the sign of the number x. In the Table 1, we give examples of the functions charactering several known algebras:

3 Conclusions

In conclusion, we have demonstrated that it is possible to construct a unified representation theory of the nonlinear deformations of the su(2), su(1,1), osp(1| 2) and sl(2) algebras. The proposed construction permits the calculation of the representations of a large number of three generator algebras, admitting a normal order and having a two generator subalgebra. The deformed algebras show features which can not be seen in the ordinary or q-deformed algebras. We have shown that these algebras have Casimir like operators, obeying deformed commutation relations with the generators of the algebra. The main result of this study is that the construction of the Verma modules, and hence the construction of the respective representations, is facilitated by the existence of a chain of subalgebras with normal ordering. The notion of the normal ordering could be the leading idea for generalisations of this construction in the cases of algebras with more than three generators.

Table 1: Characteristic functions of the several algebras.

algebra	$G(z)$	s	$f(z)$	$\Phi(\eta, m)$
$\mathrm{U}_q(\mathrm{su}(2))$	$z+1$	1	$-[2z]$	$[m]\,[-2\eta-m+1]$
$\mathrm{U}_q(\mathrm{su}(1,1))$	$z+1$	1	$[2z]$	$[m]\,[2\eta+m-1]$
$\mathrm{U}_q(\mathrm{osp}(1 \mid 2))$	$z+1/2$	-1	$-\frac{1}{4}[2z]$	$\frac{q^{\frac{1}{2}}}{4(1+q)}\Big((-1)^m\left[q^{2\eta-\frac{1}{2}}\right]$ $-\left[q^{2\eta+m-\frac{1}{2}}\right]\Big)$
$\mathcal{A}(2,1)$ [16]	$qz-1$	1	$2z(1+(1-q)z)$	$(1-q^{2m})\left(\eta+\frac{1}{1-q}\right)$ $\left(\eta-\frac{q+q^2+\cdots+q^{m-1}}{1+q^m}\right)$
$\mathcal{A}^{+}(3,1)$ [17]	$qz-1$	1	$2z(1-(1-q)^2z^2)$	$\rho(\eta)-$ $\rho\left(q^m\eta-\frac{1-q^m}{1-q}\right)$
deformed $\mathrm{U}(\mathrm{osp}(1, \mid 2))$	$1+z$	-1	$f(z)$=polynomial	$\Phi(\eta, m)$
$\mathrm{W}_3^{(2)}$[27, 28]	$2+z$	1	$-(z^2+c)$	$-m\eta^2-2m(m-1)\eta-$ $m\Big(\frac{4}{3}m^2-2m+c+\frac{2}{3}\Big)$
deformed $\mathrm{U}(\mathrm{su}(2))$[20]	$1+z$	1	$\phi(z(z-1))-\phi(z(z+1))$	$\phi((\eta+m)(\eta+m-1))-$ $\phi(\eta(\eta-1))$
polynomial sl(2) [13]	$1+z$	1	$z^n-(z+1)^n$	$(\eta+m)^n-\eta^n$

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